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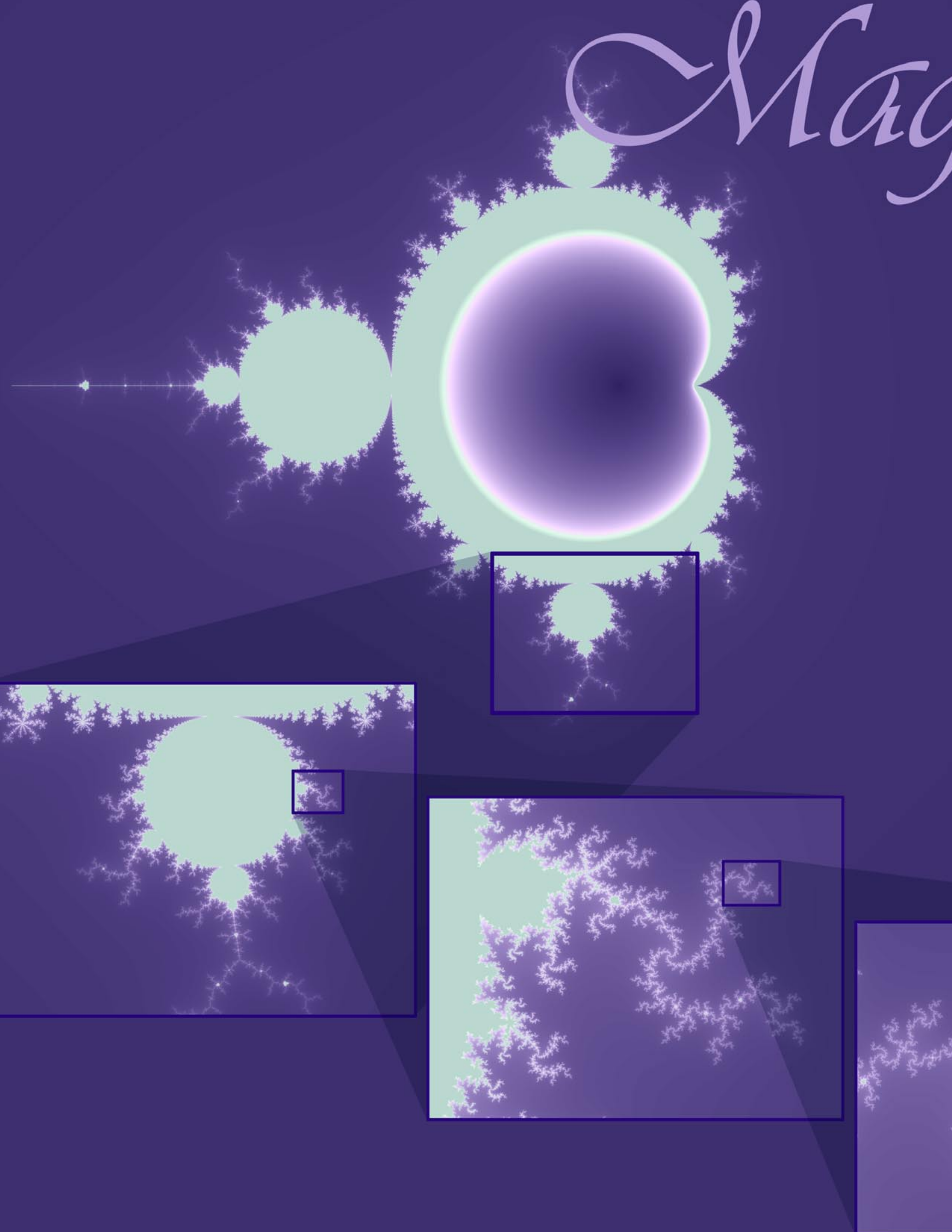


gnificent FRACTALS

Of Measures, Monsters, and Mandelbrot

By Lucy Wang and Lea Oksman

Ever notice how all your rough days are similar? There's a reason!

Bizarre Dimensions and Innumerable Clandestine Mirrors

Fractals, once considered mathematical oddities, have now become a staple of popular culture. Today, we see them everywhere from desktops to art galleries. These psychedelic patterns mesmerize the eyes and intrigue the mind. Yet it is difficult to describe exactly what makes these images so fantastic. Fractals can be simultaneously infinitely complex and amazingly simple. They appear completely alien and bizarre, yet have a natural, almost organic, appeal.

Fractals include an incredibly diverse range of figures, but share one crucial trait: self-similarity. The shape of a fractal appears at every level of magnification throughout the fractal. In a simple example, such as the Sierpinski Triangle, a fractal is created from an equilateral triangle. The midpoints of the sides are connected, dividing the triangle into four equilateral triangles, of which the center triangle is removed. As this process is repeated infinitely, each of the three main sub-triangles constitutes a smaller complete copy of the entire fractal. The three sub-triangles of those triangles, in turn, constitute an even smaller, but still complete, copy of the entire fractal. Though not all fractals have perfect self-similarity, all fractals contain a version of the entire fractal within themselves.

Actually, self-similarity is not the most bizarre property of fractals. The word "fractal" is a reference to the broken or fractured nature of these shapes. A surprising consequence of their shape is that fractals can have fractional, and not only whole number, dimensions. That is, fractals can be 2.3-dimensional or 1.2-dimensional. In the example above, the original triangle is 2-dimensional, but as more and more pieces of the triangle get removed, the image appears to become more 1-dimensional. As the pieces being removed get smaller and smaller, the entire image converges to Sierpinski's Triangle, which has a dimension of around 1.5850. On the other hand, in the case of the Koch Snowflake, kinks are added to a line in an iterative process. Since the process is infinite, infinite kinks can be added to the line, giving it an infinite length. In this process, the "line" changes from one-dimensional to 1.2618-dimensional. The result is that from extremely simple processes, strange and complex images are formed.

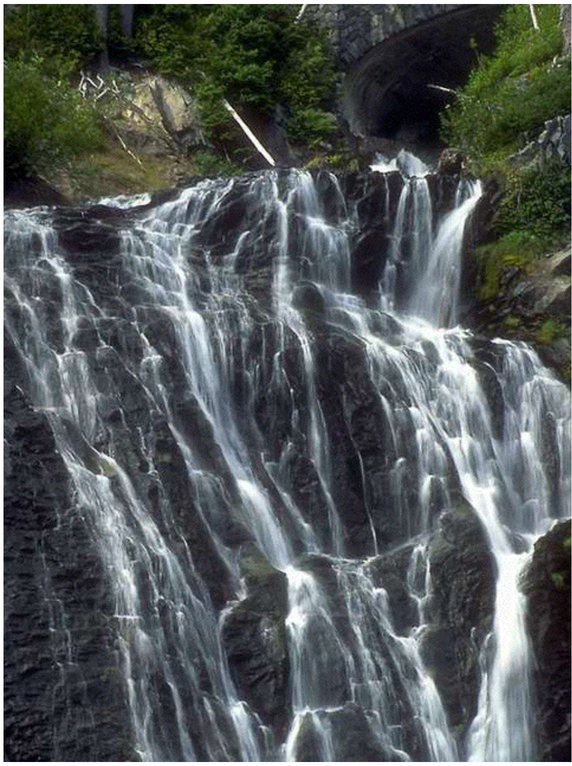
Math Monsters Escape Into the Real World

The notions of non-integer dimensions and infinite self-repetition seem rather strange – and not only to laymen. Dr. Peter Clark of the University of St. Andrews recently noted that "mathematicians [once] set about barring these monsters and they were set aside as too strange to be of interest."

The emergence of fractals as objects "of interest" began when a rather curious character appeared on the mathematical scene in France. Benoit Mandelbrot cherished one unique skill of his above all others: his "eye." While at the time mathematics students were "told to forget about visual insights in math because they were exhausted," Mandelbrot ignored this advice and proceeded to draw on nature for sources of mathematical inspiration. He made two important observations. First, Mandelbrot noticed that some natural phenomena exhibit self-similarity – for instance, a magnified section of a rock formation shows stratification in a pattern similar to that of the entire rock. With the same pattern repeated over and over again at increasing magnifications, these phenomena grow indescribably complex.

Second, Mandelbrot paid attention precisely to this "indescribability," our inability to measure a property so pervasive and obvious on an intuitive level – roughness. He realized that there was something in common among the roughness of a coastline, a curve depicting price changes over time, the pattern traced by a particle undergoing Brownian motion and infinite other examples. What struck Mandelbrot was that science had no name for this phenomenon and that roughness was never studied as a single, distinct property.

"All science begins by replacing an intuitive comparison with a number," says Mandelbrot. Indeed, when he was in school, science could produce numbers as measurements of size, luminance, acidity, and pitch, but not roughness. Mandelbrot's achievement has been to fill that gap and to find a way to make this measurement.


Courtesy of Michael Frame

Natural waterfalls exhibit self-similarity.

Mandelbrot was exploring a field no one had approached before – in fact creating the field of fractal geometry. Lacking an academic tradition and authority to guide him, he started with simple phenomena. Zooming in on a coastline, one can always see more and more bumps; it is never smooth, no matter how close to it you get. There was roughness. Looking further, one notices that the bumps are similar: ignoring magnification, the Gulf of Mexico is not awfully different from a semicircular kink in the sand measuring less than an inch in diameter. There was fractal self-similarity. Now "all" Mandelbrot had to do was link the two with some solid mathematical equations.

That was precisely where the mathematical "monsters" came in. Mandelbrot stumbled upon some mathematical techniques invented earlier and for entirely different reasons, and found them applicable to his pursuits. He developed powerful mathematical conjectures, but his formal schooling having been somewhat haphazard, was unable to prove them. Yet the key was that, as he puts it, he was "very good at asking difficult questions."


Courtesy of Michael Frame

Each section of the fern is almost a complete copy of the entire fern.

With the advent of computers, answers began to emerge. In the 1960s, Mandelbrot was working at IBM in the United States. Computers at that time were huge, room-filling monsters, and access to them was highly restricted. Mandelbrot and one or two of his programming assistants set out to do the unthinkable – to make the unwieldy machines produce pictures. And pictures they did produce: elaborate images of fractals that could be rotated, sectioned, and zoomed in on. The computer became, as Mandelbrot saw it, "a tool to enhance the skill of the eye."

More Than Just Pictures

Once Mandelbrot pulled fractal-generating equations out of the mathematical dustbin, their influence began to penetrate an increasingly large number of fields of study. Since Mandelbrot's revolutionary insight, fractals have become an increasingly widely used tool in analyzing both natural and man-made systems. Artists and mathematicians alike have noticed that natural objects show self-similarity. The branches of a tree, for example, have a structure similar to the structure of the entire tree. As

Mandelbrot puts it, "clouds are not spheres," or most natural objects are more accurately modeled by fractals than by smooth Euclidean shapes.

Fractal-generated images of everything from mountains and clouds to the surface of the moon have been used to produce realistic environments in Hollywood movies, such as *Apollo 13*, *Titanic*, and *The Perfect Storm*. At the same time, fractality has been observed in patterns of stock market behavior, the statistical structure of language and music, turbulence, and even human heartbeat patterns, to name just a few things. As Michael Frame, professor of mathematics at Yale, explains, "Nature has a fractal character." Thus, fractals have become an invaluable tool in modeling the behavior of all sorts of systems.

One characteristic of fractals that has become particularly useful is the property that some "fill up" an area or volume. Fractals such as the Peano curve start with a line segment and then begin to fill up area with the line by adding infinite kinks to the line. Though it is not physically possible to conduct the process infinitely, approximating the structure of a Peano curve will result in the compaction of a long length into a small area. This is particularly useful for cellular phone technology. Instead of using a straight rod for an antenna, some cellular phone antennas are now made in fractal shapes. Fractal antennas take up less space, and thus can be placed inside the phone. Furthermore, since the length of the antenna determines the frequency to which it will be sensitive, the diverse range of lengths represented in a fractal antenna makes it sensitive to a large range of frequencies.

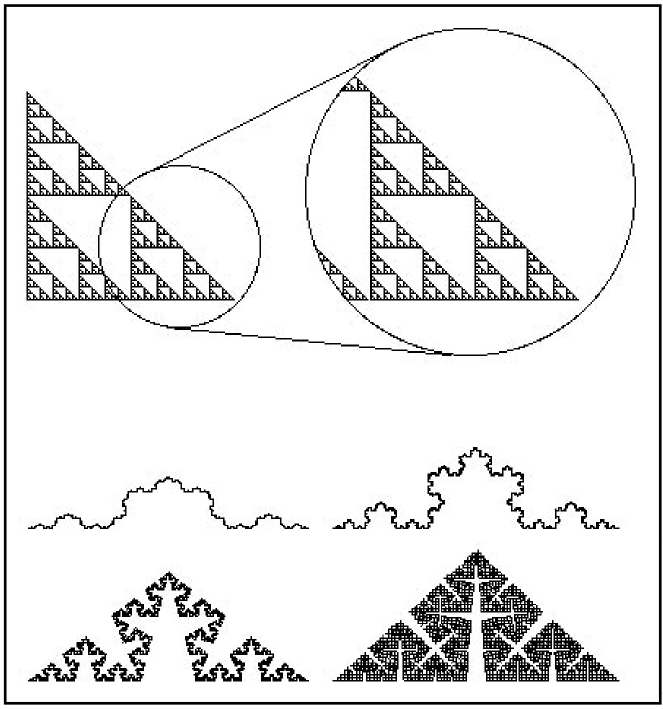
Similarly, a fractal that begins as a 2-dimensional shape, which then fills a volume, will compact a large surface area into a small volume. Since the capacitance of a system is directly proportional to the surface area, fractal capacitors are extremely effective. With fractals, powerful capacitors can be made to be extremely small. A 1-Farad capacitor, for example, can now be made smaller than the size of a pinhead.

On a more academic note, fractals have ushered in a new understanding of the great mystery of chaos. In fact, the word "chaos" has been redefined by Dr. James Yorke of the University of Maryland, who shared the 2003 Japan Prize with Mandelbrot, to mean the "nonlinear equations" that describe complex dynamic patterns such as "the motion of the planets, turbulence in water and air, [and] variations of the populations of species in ecological systems." Today, these patterns are studied mathematically using fractals.

Fractals and History

About ten years ago, John Lewis Gaddis, a Yale history professor widely known for his research on the Cold War, first became interested in fractals. His recently published book, *The Landscape of History*, discusses his thoughts on this interest. In a nutshell, Gaddis believes that in their quest for accuracy and depth, historians must study events not only across time and space, but also at different scales – from major upheavals of empires to perturbations in the lives of individual protagonists. "Until Mandelbrot came along, we never had a very good word to describe this kind of process," says Gaddis. He notes that "self-similarity is an excellent metaphor" – a new kind of vocabulary – "for historical processes." For instance, biographers have long noticed that a person's character traits tend to manifest themselves similarly at small and large scales – at home, at war, and during diplomatic negotiations. A good biographer knows and emphasizes this. Gaddis, who teaches a seminar on biography, believes one of the best ways to teach this kind of perceptiveness is to visualize this repetitive phenomenon using fractals.

Fractals resonate with history in yet another way. Consider one of the questions that brought about the interest in fractals: measuring the coastline of Britain. Because of its infinite roughness, no number that reflects the exact length of the coast can be derived. Yet without a doubt, the coast does exist and has an objective length. The same thing, according to Gaddis, happens with historical events. "Think of history as a landscape," he says. "All historians would acknowledge that there is no single, definitive account of the French Revolution"– there are so many perspectives to look from, so many facets, small and large, to explore. Nevertheless, "the French Revolution still happened," and it happened a certain way. This, Gaddis believes, is important in defying the postmodernist view that no objectivity is possible in history because bias is inevitable. Although it is almost impossible to separate the observer from his observations, we know that every historical event *has* a specific, albeit infinitely complex, objective reality.



Courtesy of Michael Frame

Above: Repeated iterations in Sierpinski's Triangle result in three copies of the entire fractal within the fractal. Below: Each third of the Koch curve is a complete copy of itself. As more and more iterations are included, the "line" begins to fill the 2-dimensional space.

The relationship of fractals to chaos also has its analogies in the study of history. The repetition of similar minute occurrences within, say, war and famine has a fractal nature, and the dynamics of such complex events are certainly chaotic in Yorke's sense of the word. Moreover, Gaddis says, good historians always understand that predictable and unpredictable phenomena coexist, that accidents determine events along with major forces – and this is precisely what mathematical studies of chaos are revealing. "It's nice to have mathematicians confirming this," he chuckles.

All of this does not mean – and Gaddis emphasizes this point – that there must be some universal equation that will explain all the events of math and history in one fell swoop, truly tying them to each other. His thoughts on fractals are an analogy, a way of organizing information. And this illuminates a very important role for the study of fractality: not only is it a new mathematical and technological tool, but it is also a new way of thinking about and approaching the world.

Fractals in the Classroom

Another promising application of fractals is in education. For the past few years, Yale has been running a summer program to train high school math teachers to teach fractals. Learning about fractals gives high school students a completely new perspective on math. Most high school math has the reputation of being a static, uninteresting abstraction unrelated to the real world. Fractals, however, mimic nature in their complexity and are intimately connected to reality.

This approach works for college students as well. Yale currently features a unique, web-based course on fractals for non-science majors. Professor Frame, who teaches the class, explains, "A lot of schools have classes on fractals in their math departments that are very technical. What makes Math 190 unique is that we emphasize the visual aspect."

Math 190 explores fractals in a variety of fields such as art, music and poetry. Students find that fractals appear in works ranging from the art of Jackson Pollock to the plays of Tom Stoppard. Though Frame initially thought there would be little interest in the class, Math 190 has attracted over 100 students every year it has been offered. Both Frame and Mandelbrot agree that Yale students are lucky to belong to a university that really gives students and professors the freedom to try unconventional things. As a result of this openness, Mandelbrot's unconventional vision of mathematics is accessible to students with different academic backgrounds and interests. Professor Frame comments that the class "often changes the way students perceive reality."

Mandelbrot appreciates this openness – and Yale in general. "If I didn't like Yale so much, I would have retired long ago," he says with a smile.

One of the reasons for this is probably the fact that academic openness was something he sorely missed in his youth. In the 1940s, when Mandelbrot was in his twenties and developing his ideas, Europe's mathematical academia was dominated by "purism"– a disregard for everything practical and applied in favor of the theoretical. "I was saved from going to college by our dear friend Hitler," he laughs, "and my revenge was to survive and learn to never neglect observations and only trust proofs."

When World War II ended, Mandelbrot came back and took graduate school entrance exams. Though he was totally unprepared for them, due to his lack of schooling, he was able to pass "by drawing figures that the teachers did not conceive were applicable."

"There is a lot more geometry in algebra than I previously realized," he comments. Despite this success, he lasted exactly one day in Ecole Normale – with his passion for everything practical and observable, the doctrine of purism was not to his taste.

"I hated it. They told me my interests were ridiculous. I said to myself, '*they* are ridiculous,' and walked out."

Office of Public Affairs/Yale University

Professor Benoit B. Mandelbrot: the mathematician with the "eye" that saw a new mathematical realm.

Mandelbrot's professors were wrong. His independent, unorthodox style of mathematical thought and acute sense of reality have enabled him to contribute to an amazingly wide array of disciplines: linguistics, computer science, economics, physics, geography, education, engineering and many others. Consider, for instance, that at Harvard he taught math, applied physics, and economics – quite a range of subjects. When asked what he is doing today, Mandelbrot smiles and says he is still working on tying everything together, writing his memoirs and another book. He describes himself as "an aging student, with many assignments." For Mandelbrot, the work never ends – like a fractal, it bends and curls onto itself, with no end and no definite beginning, always expanding to cover more and more fascinating territory.

About the Authors

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References

- O'Connor, J.J. and Robertson, E.F. *Benoit Mandelbrot.* University of St. Andrews, 1999. Available at: www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Mandelbrot.html
- http://www.math.yale.edu/mandelbrot/

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Further Reading

- Peitgen, H.O., Jurgens, H. and Saupe, D. "Chaos and Fractals: New Frontiers in Science." Springer-Verlag, 1992.
- Mandelbrot, Benoit B. *The Fractal Geometry of Nature.* New York, NY: W.H. Freeman & Company, 1982.
- Peak, David and Frame, Michael. *Chaos Under Control: The Art and Science of Complexity.* W. H. Freeman Company, 1994.
- Gaddis, John Lewis. *The Landscape of History: How Historians Map the Past.* Oxford University Press, 2002.